# WATERTOWN ARSENAL LABORATORIES

## RECTANGULAR TENSILE BAR WEAKENED BY SURFACE CRACKS
## PART I, STRESS ANALYSIS

TECHNICAL REPORT WAL TR 811.8/5

BY

OSCAR L. BOWIE



DATE OF ISSUE - MARCH 1963

AMC CODE 5011.11.838
BASIC RESEARCH IN PHYSICAL SCIENCES

D/A PROJECT 1-A-0-10501-B-010

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## ABSTRACT

The stress analysis of a rectangular tensile bar weakened by surface cracks is formulated in terms of mapping functions of a complex variable. Although an approximate solution of this problem has been previously considered by Tait, the present derivation specifies the proper load-free boundary conditions on the lateral edges of the rectangle as well as allowing for the variation of both crack length and length/width ratios of the plate. Determination of the stress function is reduced to the solution of a linear system of simultaneous equations.

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## INTRODUCTION

In the area of crack studies, a problem of practical importance is the simple geometry of a rectangular tensile specimen weakened by surface cracks. In order to apply such theories as that of Irwin[1], it is necessary to determine the stress distribution in the vicinity of the crack root. Although the behavior of the stress in such a neighborhood is known qualitatively, quantitative results require a formal solution of the stress problem as a whole. It is this matter which is considered in this report for the plane problem shown in Figure 1.

R. J. Tait[2] has considered the problem of determining the state of stress due to surface cracks in an infinite elastic strip of finite width. Unfortunately, in his analysis, load-free conditions on the strip surfaces are only partially prescribed. In fact, a mixed boundary value problem was assumed in his analysis which, in effect, is intrinsic to an infinite sheet containing a row of periodically spaced cracks.

In the present report, we shall consider surface cracks in rectangular plates of finite length and width with the proper load-free boundary conditions on the lateral edges of the rectangle. In addition to introducing the desirable modification of Tait's analysis, it is possible to study the additional parameter, length/width ratio and its effect on the stress concentration. The polynomial approximation method used by the author[3] in the analysis of an infinite plate containing radial cracks does not appear well suited to the present problem due to slowness of convergence. Thus, an alternate approximation scheme is devised.

## CONVENTIONAL COMPLEX VARIABLE FORMULATION

Muskhelishvili's[4] original complex variable statement of the plane problem in rectangular coordinates depends on the representation of Airy's stress function, $U(x,y)$, in terms of the two analytic functions, $\varphi(Z)$ and $\psi(Z)$, of the complex variable $Z=x+iy$ as

$$U(x,y) = \mathrm{Re}\left[\overline{Z}\,\varphi(Z) + \int^{Z} \psi(Z)dZ\right]. \tag{1}$$

With this representation, the stress and displacement components can be written as

$$\sigma_y + \sigma_x = 2\left[\varphi'(Z) + \overline{\varphi'(Z)}\right] = 4\,\mathrm{Re}\left[\varphi'(Z)\right], \tag{2}$$

$$\sigma_y - \sigma_x + 2i\,\tau_{xy} = 2\left[\overline{Z}\,\varphi''(Z) + \psi'(Z)\right], \tag{3}$$

$$2\mu\ (u+iv) = K\varphi(Z) - Z\ \overline{\varphi'(Z)} - \overline{\psi(Z)}, \tag{4}$$

where the prime notation denotes differentiation with respect to Z, the bars denote complex conjugates, and $\mu$ and K are material constants. A useful relation for handling stress boundary conditions is provided by

$$\varphi(Z) + Z\ \overline{\varphi'(Z)} + \overline{\psi(Z)} = i \int_0^s (X_\nu + iY_\nu)\ ds = f_1(S) + if_2(S), \tag{5}$$

where $X_\nu ds$ and $Y_\nu ds$ represent the x- and y- components of the force acting on the element of arc ds from the positive side of the normal $\nu$.

Curvilinear coordinates can be conveniently introduced by considering an auxiliary complex $\zeta$-plane along with a conformal mapping function

$$Z = \omega(\zeta). \tag{6}$$

The analyticity of (6) ensures the analyticity of the stress functions when considered as functions of the variable $\zeta$. New notation is simplified by designating $\varphi(Z) \equiv \varphi[\omega(\zeta)]$ as $\varphi(\zeta)$, etc., leading to such relationships as $\varphi'(Z) = \varphi'(\zeta)/\omega'(\zeta)$, etc.

The stresses and displacements can then be written as

$$\sigma_y + \sigma_x = 2\left[\varphi'(\zeta)/\omega'(\zeta) + \overline{\varphi'(\zeta)}/\overline{\omega'(\zeta)}\right], \tag{7}$$

$$\sigma_y - \sigma_x + 2i\ T_{xy} = 2\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\left(\frac{\varphi'(\zeta)}{\omega'(\zeta)}\right)' + \frac{\psi'(\zeta)}{\omega'(\zeta)}\right], \tag{8}$$

$$2\mu\ (u + iv) = K\ \varphi(\zeta) - \omega(\zeta)\ \overline{\varphi'(\zeta)}/\overline{\omega'(\zeta)} - \overline{\psi(\zeta)}\ . \tag{9}$$

When C corresponds to a closed contour in the Z-plane into which the unit circle $\zeta = \sigma = e^{i\theta}$ is mapped by (6), the arc parameter S can be considered as a known function of $\sigma$, hence

$$f_1\ (S) + i\ f_2\ (S) = g\ (\sigma). \tag{10}$$

The boundary condition on C (assuming the applied load is given) can thus be written as

$$\varphi(\sigma) + \omega(\sigma)\ \overline{\varphi'(\sigma)}/\overline{\omega'(\sigma)} + \overline{\psi(\sigma)} = g(\sigma). \tag{11}$$

## THE MAPPING FUNCTION

The region occupied by the rectangle with surface cracks will be denoted by $S_Z$ with boundary T in the the Z-plane (Figure 1).

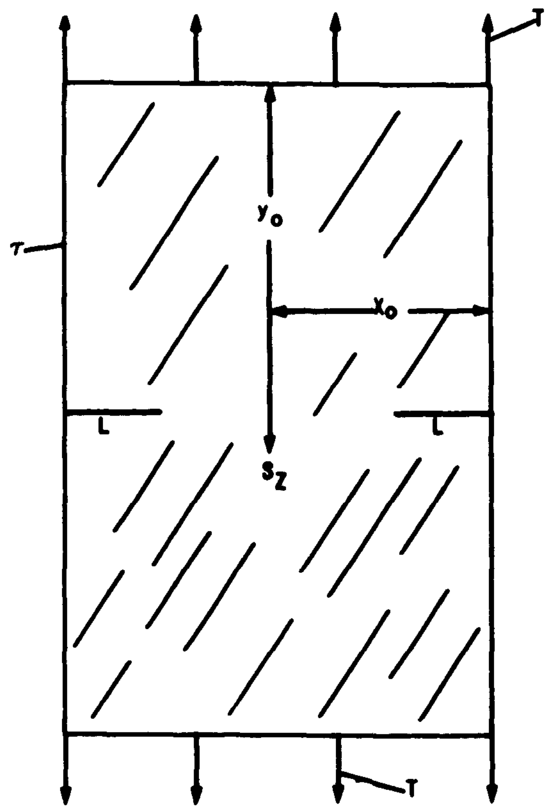


Figure I

The exact mapping function $\omega(\zeta)$ is defined as the mapping of the unit circle and its interior into the boundary $\tau$ and $S_Z$, respectively.

By a straightforward consideration of the Schwartz-Christoffel transformation, it can be seen that the exact mapping function corresponds to an appropriate branch of the function

$$Z = -i \int_0^{\zeta} (\zeta^2 + 1) \left[ (1-e^{2i\alpha}\,\zeta^2)(1-e^{-2i\alpha}\,\zeta^2)(1-e^{2i\beta}\,\zeta^2)(1-e^{-2i\beta}\,\zeta^2) \right]^{-1/2} d\zeta. \tag{12}$$

The exact mapping function has eight branch points located on the unit circle necessary to the description of the eight right angle corners. We define the real parameters in (12) such that $0<\alpha<\beta\leq\pi/2$, thus, when $\beta=\pi/2$ the circle maps into the limiting case of a rectangle. The crack roots are described by the zeroes of $\omega'(\zeta)$ on the unit circle, i.e., $\zeta=\pm i$. The ratios $L/x_o$ and $y_o/x_o$ can be varied by appropriately adjusting the parameters $\alpha$ and $\beta$.

In series form,

$$Z = \omega(\zeta) = -i \sum_{n=1}^{\infty} A_n \zeta^{2n-1} \tag{13}$$

where the $A_n$'s are real. The $A_n$'s may be obtained numerically from a straightforward recursive formulae (Appendix I) determined by comparing (12) and (13). The series (13) converges interior to the unit circle and on any interval of the unit circle not including a branch point.

Unfortunately, the rate of convergence of (13) is slow on the unit circle, thus leading to extremely tedious numerical calculation in the stress analysis. On the other hand, the procedure of polynomial approximation of the mapping function utilized in Reference 3 is plagued by the same slowness of convergence.

We adopt a different approach and define for $K\geq 1$,

$$\omega(\zeta,K)= -i \int_0^{\zeta}(\zeta^2+1)\left[(K^2-e^{2i\alpha}\zeta^2)(K^2-e^{-2i\alpha}\zeta^2)(K^2-e^{2i\beta}\zeta^2)(K^2-e^{-2i\beta}\zeta^2)\right]^{1/2} d\zeta. \tag{14}$$

Clearly $\omega(\zeta,1)$ coincides with (12). On the other hand, for $K>1$ the branch points are moved out to lie on a circle $|\zeta| = K>1$. The cusps representing the crack roots are still preserved on the unit circle at $\zeta=\pm i$. Thus, by evaluating (14) for values of $K\approx 1$ on the unit circle, increasingly accurate approximations $\tau$ are obtained as $K\rightarrow 1$. On the other hand, the convergence of the series expansions for $K>1$ will be substantially improved compared with that of (12). The form of the power series (13) is preserved in the expansion of $\omega(\zeta,K)$, thus,

$$\omega(\zeta,K) = -i \sum_{n=1}^{\infty} A_n \zeta^{2n-1}, \tag{15}$$

where the $A_n$'s are real and are now considered as functions of the parameter K as well as $\alpha$ and $\beta$.

## DETERMINATION OF THE STRESS FUNCTIONS

It is necessary to determine the two stress functions $\varphi(\zeta)$ and $\psi(\zeta)$ such that both functions are analytic for $|\zeta| < 1$ and the condition (11) is satisfied on the unit circle. Apart from mathematical elegance, little is gained by any consideration other than a direct power series argument in the present formulation. The difficulty due to the zeroes of $\omega'(\zeta)$ on the unit circle can be eliminated by writing (11) in the form

$$\overline{\omega'(\sigma)}\ \varphi(\sigma) + \omega(\sigma)\ \overline{\varphi'(\sigma)} + \overline{\omega'(\sigma)}\ \overline{\psi(\sigma)} = \overline{\omega'(\sigma)}\ g\ (\sigma). \tag{16}$$

From symmetry considerations,

$$\varphi(\zeta) = i\ T \sum_{n=1}^{\infty} a_n\ \zeta^{2n-1}, \tag{17}$$

$$\psi(\zeta) = i\ T \sum_{n=1}^{\infty} b_n\ \zeta^{2n-1}, \tag{18}$$

where the coefficients $a_n$ and $b_n$ are real. If the stress state were that of uniform tension, i.e., no surface cracks were present, then $\varphi(\zeta) = T \times \omega(\zeta)/4$ and $\psi(\zeta) = T\ \omega(\zeta)/2$. Thus, we assume

$$\varphi(\zeta) = T\ \omega(\zeta)/4 + \varphi_1\ (\zeta), \tag{19}$$

$$\psi(\zeta) = T\ \omega(\zeta)/2 + \psi_1\ (\zeta), \tag{20}$$

where

$$\varphi_1(\zeta) = i\ T \sum_{n=1}^{\infty} \alpha_n\ \zeta^{2n-1}, \tag{21}$$

$$\psi_1(\zeta) = i\ T \sum_{n=1}^{\infty} \beta_n\ \zeta^{2n-1}. \tag{22}$$

Clearly,

$$\overline{\omega'(\sigma)}\ \varphi_1\ (\sigma) + \omega(\sigma)\ \overline{\varphi_1'(\sigma)} + \overline{\omega'(\sigma)}\ \overline{\psi_1(\sigma)} =$$

$$\overline{\omega'(\sigma)}\ \left\{g(\sigma) - \frac{T}{2}\ \left[\omega(\sigma) + \overline{\omega(\sigma)}\right]\right\} = T \sum_{n=-\infty}^{\infty} d_n\ \sigma^{2n-1}. \tag{23}$$

Substitution of (21) and (22) into (23) and comparing coefficients of equal powers of $\sigma$ leads to the two sets of conditions

$$\sum_{n=1}^{\infty} A_n (2n-1) \alpha_{p+n-1} + \sum_{n=1}^{\infty} \alpha_n (2n-1) A_{p+n-1} = - d_p, \; p = 1,2,3, ---- \quad (24)$$

$$\sum_{n=p+2}^{\infty} A_n (2n-1) \alpha_{n-p-1} + \sum_{n=p+2}^{\infty} \alpha_n (2n-1) A_{n-p-1}$$

$$- \sum_{n=1}^{p+1} A_n (2n-1) \beta_{p+2-n} = - d_{-p}, \; p = 0,1,2, ----. \quad (25)$$

The set of equations 25 provides for a direct determination of the $\beta_n$'s once the $\alpha_n$'s have been found from the linear system of simultaneous equations 24.

The applied load will be expanded on the basis of the exact geometry and loading. Thus, if

$$g(\sigma) - \frac{T}{2} \left[\omega(\sigma) + \overline{\omega(\sigma)}\right] = \sum_{K=-\infty}^{\infty} C_{2K-1} \sigma^{2K-1}, \quad (26)$$

then it can easily be shown (Appendix 2), that

$$C_{2K-1} = - \frac{2iT}{\pi} \frac{\cos (2K-1)\beta}{2K-1} \sum_{n=1}^{\infty} A_n \sin (2n-1) \beta$$

$$+ \frac{2iT}{\pi} \int_{\beta}^{\pi/2} \sum_{n=1}^{\infty} A_n \sin (2n-1) \theta \sin (2K-1) \theta d\theta, \; K = 0, \pm 1, \pm 2, ---- \quad (27)$$

For computational purposes an alternate form of $C_{2K-1}$ is now presented.

$$C_{2K-1} = - \frac{2iT}{\pi} \int_{\beta}^{\pi/2} \left[X(\beta) - X(\theta)\right] \sin (2K-1) \theta \, d \theta, \quad (28)$$

where $X(\theta)$ is the real part of $\omega(e^{i\theta})$. Integration by parts yields

$$C_{2K-1} = \frac{2iT}{(2K-1)\pi} \int_{\beta}^{\pi/2} \frac{dX}{d\theta} \cos (2K-1) \theta \, d \theta. \quad (29)$$

Using (12) to compute $dX/d\theta$, it follows that

$$C_{2K-1} = \frac{2iT}{(2K-1)\pi} \int_{\beta}^{\pi/2} \frac{\cos\theta \cos(2K-1)\theta \, d\theta}{(\cos 2\alpha - \cos 2\theta)^{1/2}(\cos 2\beta - \cos 2\theta)^{1/2}}. \quad (30)$$

From the definition of the $d_n$'s in (23), it follows that

$$T\, d_p = i \sum_{n=1}^{\infty} A_n (2n-1)\, C_{2(p+n)-3}, \quad p = 0, \pm 1, \pm 2, ---- \quad (31)$$

When approximate mappings 15 are used, the loading conditions must be considered as approximated. On the other hand, as $K \to 1$ both the geometry and the applied load converge to the exact problem.

## DETERMINATION OF THE STRESS CONCENTRATION IN THE VICINITY OF THE CRACK ROOTS

The stress concentration in the vicinity of either of the crack roots is the prime aim of the analysis. If polar coordinates $(r,\gamma)$ are introduced with the origin chosen as the crack root, the stresses vary to the first order* as

$$\sigma \approx L(\gamma)/\sqrt{r}. \quad (32)$$

The determination of $L(\gamma)$ in terms of the present analysis will now be considered.

Assume that $\zeta_o$ is the point on the unit circle which maps into the crack root $Z_o$. Then $\omega'(\zeta)$ has a simple zero at $\zeta = \zeta_o$. Furthermore, $\omega(\zeta)$ is analytic in the neighborhood of $\zeta_o$ on the Riemann surface defining the mapping function, thus,

$$Z - Z_o = \sum_{n=2}^{\infty} S_n (\zeta - \zeta_o)^n. \quad (33)$$

By reversion of series,

$$\zeta - \zeta_o = \sum_{n=1}^{\infty} t_n (Z - Z_o)^{n/2}, \quad (34)$$

* $L(\gamma)$ is essentially proportional to Irwin's factor K.

where

$$S_2 t_1^2 = 1, \text{ etc.}$$

thus,

$$\zeta - \zeta_o \approx \frac{1}{\sqrt{S_2}} (Z-Z_o)^{1/2}. \tag{35}$$

If we introduce polar coordinates,

$$Z-Z_o = r\, e^{i\gamma}, \tag{36}$$

then

$$\zeta - \zeta_o \approx \frac{\sqrt{r}}{\sqrt{S_2}} e^{i\gamma/2} = \frac{\sqrt{2r}}{\sqrt{\omega''(\zeta_o)}} e^{i\gamma/2}. \tag{37}$$

In order to utilize these approximations to examine the stress field local to the notch root, it is clear from (7) and (8) that the character of $\varphi(\zeta)$ and $\psi(\zeta)$ in the neighborhood of $\zeta = \zeta_o$ is required. If we define

$$\psi(\zeta) = -\overline{\varphi}(1/\zeta) - \overline{\omega}(1/\zeta)\, \varphi'(\zeta)/\omega'(\zeta) \tag{38}$$

where

$$\overline{f}(1/\zeta) \equiv \overline{f(1/\overline{\zeta})}, \tag{39}$$

and substitute into (11), it can be shown that $\varphi(\zeta)$ is analytic in the neighborhood of $\zeta = \zeta_o$ whereas $\psi(\zeta)$ has a simple pole in this neighborhood.

It follows fairly directly from (7) that

$$\sigma_x + \sigma_y \approx \mathrm{Rl}\left\{\frac{2\sqrt{2}\,[\varphi'(\zeta_o)]}{\sqrt{\omega''(\zeta_o)}\,\sqrt{r}} e^{-i\gamma/2}\right\}. \tag{40}$$

From (8), it follows from a somewhat more difficult calculation, that

$$\sigma_y - \sigma_x + 2i\tau_{xy} \approx \frac{\sqrt{2}\, e^{-i\gamma/2}}{\sqrt{\omega''(\zeta_o)\, r}} \left[\frac{\overline{\varphi'(\zeta_o)}}{\zeta_o^2} - \frac{\varphi'(\zeta_o)}{2\zeta_o^4} - \frac{\varphi'(\zeta_o)}{2} e^{-2i\gamma}\right]. \tag{41}$$

In the calculation of (40) and (41), the key quantity is $\varphi'(\zeta_o)$. Whereas the calculation of $\omega''(\zeta_o)$ can be made directly from the closed form of the mapping function, the determination of $\varphi'(\zeta_o)$ requires the solution of the system of equations 24.

## SUMMARY

The stress analysis for a rectangular tensile bar weakened by surface cracks has been presented. A mapping function which permits variation of the crack depth and length/width ratio of the plate is utilized. Determination of the stress field in the neighborhood of the crack root requires the calculation of $\varphi'(\zeta_o)$ where $\zeta_o$ is the point on the unit circle corresponding to the crack root. A procedure for systematically approximating $\varphi'(\zeta_o)$ is suggested by the limiting process as $K \rightarrow 1$ in the mapping function.

Calculations are underway for pertinent values of the parameters and the results will be presented in a subsequent report.

## APPENDIX I

### RECURSIVE FORMULAE FOR THE MAPPING COEFFICIENTS

From (14),

$$\omega'(\zeta,K) = -i\ (\zeta^2+1)/(\zeta^4-2K^2\zeta^2\cos 2\alpha+K^4)^{1/2}(\zeta^4-2K^2\zeta^2\cos 2\beta+K^4)^{1/2}.$$

Let

$$\omega'(\zeta,K) = -i\sum_{n=1}^{\infty} M_n\ \zeta^{2n-2}.$$

Then,

$$-(\zeta^2+1)^2 = -\left\{\zeta^8+\zeta^6K^2(-2\cos 2\beta-2\cos 2\alpha)\right.$$

$$+\ \zeta^4(2K^4+4K^4\cos 2\alpha\cos 2\beta)+\zeta^2(-2K^6\cos 2\alpha-2K^6\cos 2\beta)$$

$$\left.+\ K^8\right\}\left\{\sum_{n=1}^{\infty} M_n\ \zeta^{2n-2}\right\}^2.$$

Let

$$2\epsilon_1 = -2\cos 2\alpha-2\cos 2\beta$$

$$2\epsilon_2 = 2+4\cos 2\alpha\cos 2\beta$$

$$C_2 = M_1^2$$

$$C_3 = 2M_1M_2$$

$$C_4 = 2M_1M_3+M_2^2$$

$$C_5 = 2M_1M_4+2M_2M_3 \quad \text{etc.}$$

Then,

$$(\zeta^4+2\zeta^2+1) = \left[\zeta^8+2K^2\epsilon_1\zeta^6+2K^4\epsilon_2\zeta^4+2K^6\epsilon_1\zeta^2+K^8\right]$$

$$\left[C_2+C_3\zeta^2+C_4\zeta^4+\text{-----}\right].$$

Equating coefficients of equal powers of $\zeta$ yields

$$C_2 K^8 = 1$$

$$K^8 C_3 + 2 K^6 \epsilon_1 C_2 = 2$$

$$K^8 C_4 + 2 K^6 \epsilon_1 C_3 + 2 K^4 \epsilon_2 C_2 = 1$$

$$K^8 C_5 + 2 K^6 \epsilon_1 C_4 + 2 K^4 \epsilon_2 C_3 + 2K^2 \epsilon_1 C_2 = 0$$

$$K^8 C_6 + 2 K^6 \epsilon_1 C_5 + 2 K^4 \epsilon_2 C_4 + 2 K^2 \epsilon_1 C_3 + C_2 = 0$$

etc.

Thus,

$$M_1 = -1/K^4$$

$$M_2 = -K^4 C_3/2$$

$$M_3 = -K^4 \left[ C_4/2 - M_2{}^2/2 \right]$$

$$M_4 = -K^4 \left[ C_5/2 - M_2 M_3 \right]$$

etc.

Thus, in the expansion

$$\omega(\zeta, K) = - i \ \Sigma A_n \zeta^{2n-1},$$

$$A_n = M_n/(2n-1).$$

## APPENDIX II

## EXPANSION OF THE LOADING FUNCTION

It will be recalled that

$$g(\sigma) = i \int_0^S (X_\nu + i Y_\nu)\, dS.$$

We define

$$g(\sigma) = TX, \quad \left[-X_0 \le X \le 0,\ y = y_0\right]$$

$$= -TX_0, \left[X = -X_0,\ 0 \le y \le y_0\right]$$

etc.

Thus,

$$g(\sigma) - \frac{T}{2}\left[\omega(\sigma) + \overline{\omega(\sigma)}\right] = 0, \quad 0 \le \theta \le \beta$$

$$= -TX_0 - \frac{T}{2}\left[\omega(\sigma) + \overline{\omega(\sigma)}\right], \ \beta \le \theta \le \pi - \beta$$

$$= 0, \ \pi - \beta \le \theta \le \pi + \beta,$$

$$= TX_0 - \frac{T}{2}\left[\omega(\sigma) + \overline{\omega(\sigma)}\right], \left[\beta + \pi \le \theta \le 2\pi - \beta\right],$$

$$= 0, \ 2\pi - \beta \le \theta \le 2\pi.$$

If we set

$$g(\sigma) - \frac{T}{2}\left[\omega(\sigma) + \overline{\omega(\sigma)}\right] = \sum_{k=-\infty}^{\infty} C_K \sigma^K,$$

then

$$C_K = \frac{T}{2\pi} \int_{\beta}^{\pi-\beta} \left\{ -X_o - \frac{1}{2} \left[ \omega(\sigma) + \overline{\omega(\sigma)} \right] \right\} e^{-iK\theta} \, d\theta$$

$$+ \frac{T}{2\pi} \int_{\pi+\beta}^{2\pi-\beta} \left\{ X_o - \frac{1}{2} \left[ \omega(\sigma) + \omega(\sigma) \right] \right\} e^{-iK\theta} \, d\theta.$$

It follows from symmetry that

$$C_K = 0, \quad K \text{ even}$$

$$C_K = \frac{T}{\pi} \int_{\beta}^{\pi-\beta} \left\{ -X_o - \frac{1}{2} \left[ \omega(\sigma) + \overline{\omega(\sigma)} \right] \right\} e^{-iK\theta} \, d\theta, \quad K = \pm 1, \pm 3, ----.$$

Again from symmetry,

$$C_K = - \frac{2iT}{\pi} \int_{\beta}^{\pi/2} \left\{ -X_o - \frac{1}{2} \left[ \omega(\sigma) + \omega(\sigma) \right] \right\} \sin K\theta \, d\theta, \; K = \pm 1, \pm 3, ---.$$

Finally,

$$C_{2K} = 0, \quad K = 0, \pm 1, \pm 2, ----$$

$$C_{2K-1} = - \frac{2iT}{\pi} \, \frac{\cos (2K-1) \beta}{2K-1} \sum_{n=1}^{\infty} A_n \sin (2n-1) \beta$$

$$+ \frac{2iT}{\pi} \int_{\beta}^{\pi/2} \sum_{n=1}^{\infty} A_n \sin (2n-1) \theta \sin (2K-1) \theta d\theta, \quad K=0, \pm 1, \pm 2, ---.$$

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The stress analysis of a rectangular tensile bar weakened by surface cracks is formulated in terms of mapping functions of a complex variable. Although an approximate solution of this problem has been previously considered by Tait, the present derivation specifies the proper load-free boundary conditions on the lateral edges of the rectangle as well as allowing for the variation of both crack length and length/width ratios of the plate. Determination of the stress function is reduced to the solution of a linear system of simultaneous equations.

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